Physics 207, Lecture 14, Oct. 23

Agenda: Chapter 10, Finish, Chapter 11, Just Start

- Chapter 10:
  - ❖ Moments of Inertia
  - ❖ Parallel axis theorem
  - ❖ Torque
  - ❖ Energy and Work
- Chapter 11
  - ❖ Vector Cross Products
  - ❖ Rolling Motion
  - ❖ Angular Momentum

Assignment: For Wednesday reread Chapter 11, Start Chapter 12

- WebAssign Problem Set 5 due Tuesday
- Problem Set 6, Ch 10-79, Ch 11-17,23,30,35,44abdef Ch 12-4,9,21,32,35

Moment of Inertia and Rotational Energy

- So $K = \frac{1}{2}I\omega^2$ where $I = \sum_i m_i r_i^2$
- Notice that the moment of inertia I depends on the distribution of mass in the system.
  - ❖ The further the mass is from the rotation axis, the bigger the moment of inertia.
- For a given object, the moment of inertia depends on where we choose the rotation axis (unlike the center of mass).
- In rotational dynamics, the moment of inertia I appears in the same way that mass *m* does in linear dynamics !

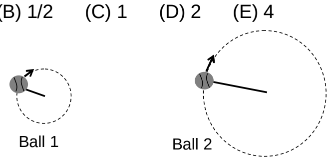
Lecture 14, *Exercise 1*
Rotational Kinetic Energy

- We have two balls of the same mass. Ball 1 is attached to a 0.1 m long rope. It spins around at 2 revolutions per second. Ball 2 is on a 0.2 m long rope. It spins around at 2 revolutions per second.
- What is the ratio of the kinetic energy of Ball 2 to that of Ball 1 ?

$K = \frac{1}{2}I\omega^2$

$I = \sum_i m_i r_i^2$

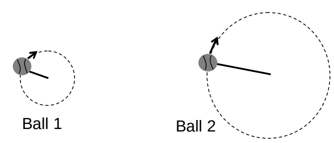
(A) 1/ (B) 1/2 (C) 1 (D) 2 (E) 4



Lecture 14, *Exercise 1*
Rotational Kinetic Energy

- $K_2/K_1 = ½ m \omega r_2^2 / ½ m \omega r_1^2 = 0.2^2 / 0.1^2 = 4$
- What is the ratio of the kinetic energy of Ball 2 to that of Ball 1 ?

(A) 1/ (B) 1/2 (C) 1 (D) 2 **(E) 4**



Lecture 14, *Exercise 2*
Moment of Inertia

- A triangular shape is made from identical balls and identical rigid, massless rods as shown. The moment of inertia about the *a*, *b*, and *c* axes is I_a, I_b, and I_c respectively.

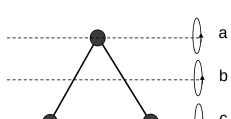
$I = \sum_i m_i r_i^2$

- ❖ Which of the following is correct:

(A) $I_a > I_b > I_c$

(B) $I_a > I_c > I_b$

(C) $I_b > I_a > I_c$



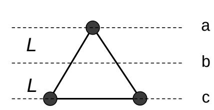
Lecture 14, *Exercise 2*
Moment of Inertia

- $I_a = 2\ m\ (2L)^2$ $I_b = 3\ m\ L^2$ $I_c = m\ (2L)^2$
- Which of the following is correct:

(A) $I_a > I_b > I_c$

(B) $I_a > I_c > I_b$

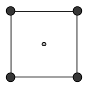
(C) $I_b > I_a > I_c$



Calculating Moment of Inertia...

- For a discrete collection of point masses we find:

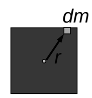
$$I = \sum_{i=1}^{N} m_i r_i^2$$



- For a continuous solid object we have to add up the mr^2 contribution for every infinitesimal mass element *dm*.

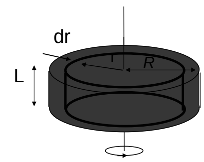
❖ An integral is required to find I :

$$I = \int r^2 dm$$



Moments of Inertia

- Some examples of I for solid objects:

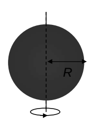


Solid disk or cylinder of mass *M* and radius *R*, about perpendicular axis through its center.

$I = ½ M R^2$

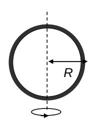
Moments of Inertia...

- Some examples of I for solid objects:



Solid sphere of mass *M* and radius *R*, about an axis through its center.

I = 2/5 M R2



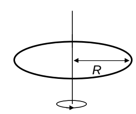
Thin spherical shell of mass *M* and radius *R*, about an axis through its center.

Use the table…

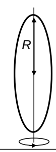
See Table 10.2, Moments of Inertia

Moments of Inertia

- Some examples of I for solid objects:



Thin hoop (or cylinder) of mass *M* and radius *R*, about an axis through it center, perpendicular to the plane of the hoop is just MR^2



Thin hoop of mass *M* and radius *R*, about an axis through a diameter.

Use the table…

Parallel Axis Theorem

- Suppose the moment of inertia of a solid object of mass *M* about an axis through the center of mass is known and is said to be I_{CM}
- The moment of inertia about an axis parallel to this axis but a distance *R* away is given by:

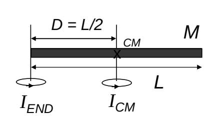
$$I_{PARALLEL} = I_{CM} + MR^2$$

- So if we know I_{CM} , one can calculate the moment of inertia about a parallel axis.

Parallel Axis Theorem: Example

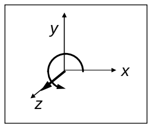
- Consider a thin uniform rod of mass *M* and length *D*. What is the moment of inertia about an axis through the end of the rod?

$$I_{PARALLEL} = I_{CM} + MD^2$$

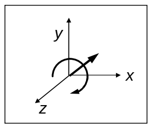


Direction of Rotation:

- In general, the rotation variables are vectors (have magnitude and direction)
- If the plane of rotation is in the *x-y* plane, then the convention is
  - ❖ CCW rotation is in the + *z* direction

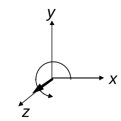


  - ❖ CW rotation is in the - *z* direction

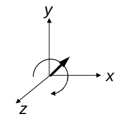


Direction of Rotation: The Right Hand Rule

- To figure out in which direction the rotation vector points, curl the fingers of your right hand the same way the object turns, and your thumb will point in the direction of the rotation vector !



- In Serway the z-axis to be the rotation axis as shown.
  - ❖ $\theta = \theta_z$
  - ❖ $\omega = \omega_z$
  - ❖ $\alpha = \alpha_z$



- For simplicity the subscripts are omitted unless explicitly needed.

Newton's 2nd law: Rotation

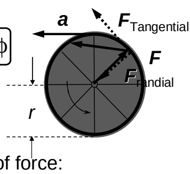
- Linear dynamics: $$\vec{F} = m\vec{a}$$
- Rotational dynamics: $$\tau_z = I_z \alpha_z$$

Where τ is referred to as "torque" and τ_z is the component along the z-axis

$$\vec{\tau} = \tau_x \hat{i} + \tau_y \hat{j} + \tau_z \hat{k}$$

Rotational Dynamics: What makes it spin?

$\tau_{TOT} = I\,\alpha = |F_{Tang}|\, r = |F|\,|r| \sin\phi$



- This is the rotational version of $F_{TOT} = ma$
- Torque is the rotational equivalent of force:
  - The amount of "twist" provided by a force.
  - A big caveat (!) – Position of force vector matters (**r**)
- Moment of inertia I is the rotational equivalent of mass.
  - If I is big, more torque is required to achieve a given angular acceleration.
- Torque has units of kg m²/s² = (kg m/s²) m = N m

Newton's 2nd law: Rotation
Vector formulation

- Linear dynamics: $$\vec{F} = m\vec{a}$$
- Rotational dynamics:

$$\vec{\tau} = I\vec{\alpha}$$

(where I is axis dependent)

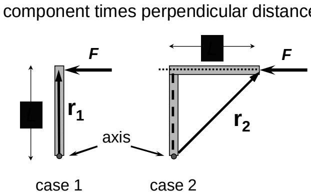
$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{where} |\tau| = |r|\,|F| \sin\theta$$

once we define the "vector cross product"

Lecture 14, *Exercise 3*
Torque

- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.
- Torque requires ***F***, ***r*** and sin θ <u>or</u> translation along tangent or the tangential force component times perpendicular distance

(A) case 1
(B) case 2
(C) same



Lecture 14, *Exercise 3*
Torque

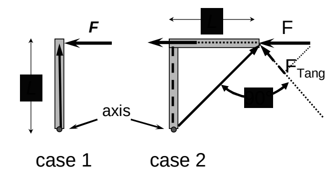
- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.
- Remember torque requires ***F***, ***r*** and sin ϕ

or the tangential force component times perpendicular distance

(A) case 1

(B) case 2

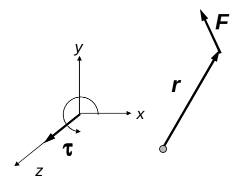
(C) same



See text: 11.2

Torque (as a vector) and the Right Hand Rule:

- The right hand rule can tell you the direction of torque:
  - ❖ Point your hand along the direction from the axis to the point where the force is applied.
  - ❖ Curl your fingers in the direction of the force.
  - ❖ Your thumb will point in the direction of the torque.



See text: 11.2

The Vector Cross Product

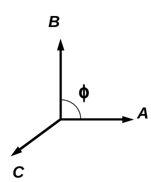
- The can obtain the vectorial nature of torque in compact form by defining a "vector cross product".
  - ❖ The cross product of two vectors is another vector:

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}$$

- The length of ***C*** is given by:

$$|\boldsymbol{C}| = |\boldsymbol{A}|\,|\boldsymbol{B}|\, \sin\phi$$

- The direction of ***C*** is perpendicular to the plane defined by ***A*** and ***B***, and in the direction defined by the right-hand rule.



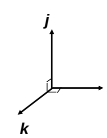
The Cross Product

- The cross product of unit vectors:

$\boldsymbol{i} \times \boldsymbol{i} = 0$ $\quad \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ $\quad \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$

$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$ $\quad \boldsymbol{j} \times \boldsymbol{j} = 0$ $\quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$

$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$ $\quad \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$ $\quad \boldsymbol{k} \times \boldsymbol{k} = 0$



$$\boldsymbol{A} \times \boldsymbol{B} = (A_X \boldsymbol{i} + A_Y \boldsymbol{j} + A_z \boldsymbol{k}) \quad \times \quad (B_X \boldsymbol{i} + B_Y \boldsymbol{j} + B_z \boldsymbol{k})$$

$$= (A_X B_X\ \boldsymbol{i} \times \boldsymbol{i} + A_X B_Y\ \boldsymbol{i} \times \boldsymbol{j} + A_X B_Z\ \boldsymbol{i} \times \boldsymbol{k})$$
$$+ (A_Y B_X\ \boldsymbol{j} \times \boldsymbol{i} + A_Y B_Y\ \boldsymbol{j} \times \boldsymbol{j} + A_Y B_Z\ \boldsymbol{j} \times \boldsymbol{k})$$
$$+ (A_Z B_X\ \boldsymbol{k} \times \boldsymbol{i} + A_Z B_Y\ \boldsymbol{k} \times \boldsymbol{j} + A_Z B_Z\ \boldsymbol{k} \times \boldsymbol{k})$$

The Cross Product

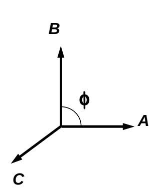
- Cartesian components of the cross product:

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$$

$$C_X = A_Y B_Z - B_Y A_Z$$

$$C_Y = A_Z B_X - B_Z A_X$$

$$C_Z = A_X B_Y - B_X A_Y$$



Note: $\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{B}$

Torque & the Cross Product:

- So we can define torque as:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

$$|\tau| = |\boldsymbol{r}|\,|\boldsymbol{F}|\, \sin\phi$$

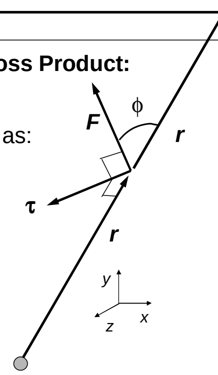
or

$$\tau_X = y F_Z - z F_Y$$

$$\tau_Y = z F_X - x F_Z$$

$$\tau_Z = x F_Y - y F_X$$

use whichever works best

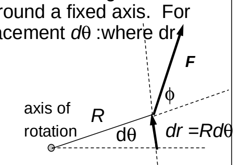


Work (in rotational motion)

- Consider the work done by a force ***F*** acting on an object constrained to move around a fixed axis. For an infinitesimal angular displacement $d\theta$:where $dr = R\,d\theta$

$dW = F_{Tangential}\,dr$

$dW = (F_{Tangential}\,R)\,d\theta$



$dW = \tau\,d\theta$ (and with a constant torque)

- We can integrate this to find: $\boxed{W = \tau\,\theta} = \tau\,(\theta_f - \theta_i)$
- Analogue of $W = \boldsymbol{F} \cdot \Delta\boldsymbol{r}$
- W will be negative if τ and θ have opposite sign !

Work & Kinetic Energy:

- Recall the Work Kinetic-Energy Theorem: $\Delta K = W_{NET}$
- This is true in general, and hence applies to rotational motion as well as linear motion.
- So for an object that rotates about a fixed axis:

$$\Delta K = \tfrac{1}{2}\,\mathrm{I}\left(\omega_f^2 - \omega_i^2\right) = W_{NET}$$

Newton's 2nd law: Rotation

- Linear dynamics: $$\vec{F} = m\vec{a}$$
- Rotational dynamics: $$\vec{\tau} = I\vec{\alpha} = \vec{r} \times \vec{F}$$

Where τ is referred to as "torque" and I is axis dependent (in Phys 207 we specify this axis and reduce the expression to the z component).

Lecture 14, *Exercise 4*
Rotational Definitions

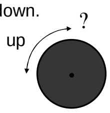
- A goofy friend sees a disk spinning and says "Ooh, look! There's a wheel with a negative ω and with antiparallel ω and α!"
- Which of the following is a true statement about the wheel?

(A) The wheel is spinning counter-clockwise and slowing down.

(B) The wheel is spinning counter-clockwise and speeding up.

(C) The wheel is spinning clockwise and slowing down.

(D) The wheel is spinning clockwise and speeding up



Lecture 15, *Exercise 4*
Work & Energy

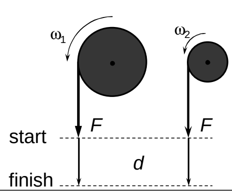
- Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same linear distance.
  Disk 1 has a bigger radius, but both are identical material (i.e. their density ρ = M/V is the same). Both disks rotate freely around axes though their centers, and start at rest.
  - ❖ Which disk has the biggest angular velocity after the pull?

$W = \tau\,\theta = F\,d = ½\,\mathrm{I}\,\omega^2$

(A) Disk 1

(B) Disk 2

(C) Same



Lecture 15, *Exercise 4*
Work & Energy

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  Disk 1 has a bigger radius, but both are identical material (i.e. their density ρ = M/V is the same). Both disks rotate freely around axes though their centers, and start at rest.
  - ❖ Which disk has the biggest angular velocity after the pull?

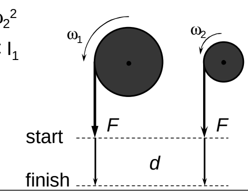
$W = F\,d = ½\,I_1\,\omega_1^2 = ½\,I_2\,\omega_2^2$

$\omega_1 = (I_2 / I_1)^{½}\ \ \omega_2$ and $I_2 < I_1$

(A) Disk 1

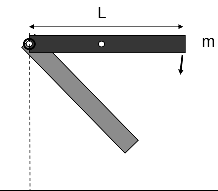
(B) Disk 2

(C) Same



Example: Rotating Rod

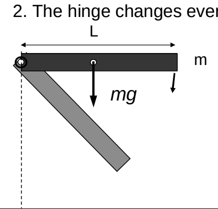
- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless pin passing through one end as in the Figure. The rod is released from rest in the horizontal position. What is
  - (A) its angular speed when it reaches the lowest point ?
  - (B) its initial angular acceleration ?
  - (C) initial linear acceleration of its free end ?



See example 10.14

Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is
  - (B) its initial angular acceleration ?
  - 1. For forces you need to locate the <u>Center of Mass</u>
  - CM is at L/2 (halfway) and put in the Force on a FBD
  - 2. The hinge changes everything!



$\Sigma \boldsymbol{F} = 0$ occurs <u>only</u> at the hinge

but $\tau_z = I\ \alpha_z = r\ F \sin 90°$

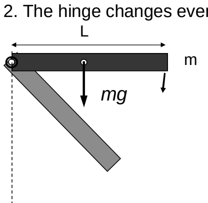
at the center of mass and

$(I_{CM} + m(L/2)^2)\ \alpha_z = (L/2)\ mg$

and <u>solve</u> for α_z

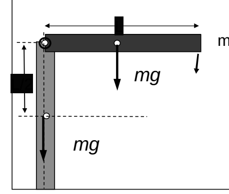
Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is
  - (C) initial linear acceleration of its free end ?
  - 1. For forces you need to locate the <u>Center of Mass</u>
  - CM is at L/2 (halfway) and put in the Force on a FBD
  - 2. The hinge changes everything!



Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is
  - (A) its angular speed when it reaches the lowest point ?
  - 1. For forces you need to locate the Center of Mass
  - CM is at L/2 (halfway) and use the Work-Energy Theorem
  - 2. The hinge changes everything!



Connection with CM motion

- If an object of mass *M* is moving linearly at velocity V_{CM} *without* rotating then its kinetic energy is

$$K_T = \tfrac{1}{2} M V_{CM}^2$$

- If an object of moment of inertia I_{CM} is rotating *in place* about its center of mass at angular velocity ω then its kinetic energy is

$$K_R = \tfrac{1}{2} I_{CM} \omega^2$$

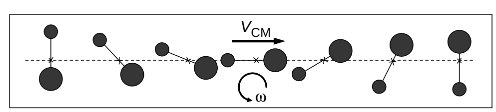
- What if the object is both moving linearly and rotating?

$$K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M V_{CM}^2$$

Connection with CM motion...

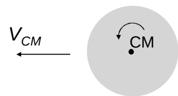
- So for a solid object which rotates about its center of mass and whose CM is moving:

$$K_{TOT} = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M V_{CM}^2$$



Rolling Motion

- Now consider a cylinder rolling at a constant speed.



The cylinder is rotating about CM and its CM is moving at constant speed (V_{CM}). Thus its total kinetic energy is given by :

$$K_{TOT} = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M V_{CM}^2$$

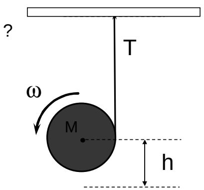
Lecture 14, *Example:* The YoYo

- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.

(1) What is the angular acceleration?

(2) What will be the linear velocity of the center of mass after it falls h meters?

(3) What is the tension on the cord ?

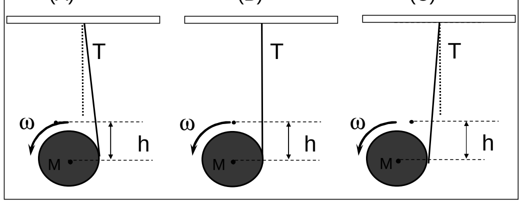


Lecture 14, *Example:* The YoYo

- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.
- Conceptual Exercise:

  Which of the following pictures correctly represents the yoyo after it falls a height h?

(A) (B) (C)

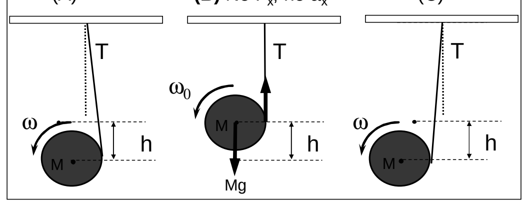


Lecture 14, *Example:* The YoYo

- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.
- Conceptual Exercise:

  Which of the following pictures correctly represents the yoyo after it falls a height h?

(A) **(B)** No F_x, no a_x (C)



Lecture 14, *Example:* The YoYo

- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.

(1) What is the angular acceleration?

(2) What will be the linear velocity of the center of mass after it falls h meters?

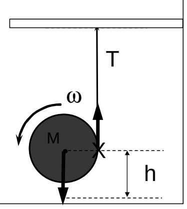
(3) What is the tension on the cord ?

Choose a point and calculate the torque

$\Sigma\tau = \; I\,\alpha_z \; = Mg\,R + T0$

$(\tfrac{1}{2} MR^2 + MR^2)\,\alpha_z = Mg\,R$

$\alpha_z = Mg\,/(3/2\,MR) = 2\,g\,/\,(3R)$



Lecture 14, *Example:* The YoYo

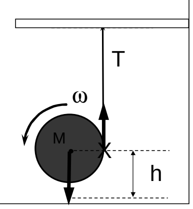
- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.

(1) What is the angular acceleration?

(2) What will be the linear velocity of the center of mass after it falls h meters?

(3) What is the tension on the cord ?

Can use kinetics or <u>work energy</u>



Lecture 14, *Example:* The YoYo

- A solid uniform disk yoyo of radium R and mass M starts from rest, unrolls, and falls a distance h.
  - (1) What is the angular acceleration?
  - (2) What will be the linear velocity of the center of mass after it falls h meters?
  - (3) What is the tension on the cord ?

$a_{CM} = \alpha_z R = -2g/3$

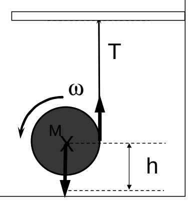
$Ma_{CM} = - 2Mg/3 = T – Mg$

$T = Mg/3$

or from torques

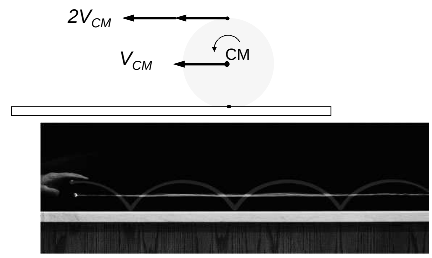
$I \alpha_{z'} = TR = ½ MR^2 (2g/3R)$

$T = Mg/3$



Rolling Motion

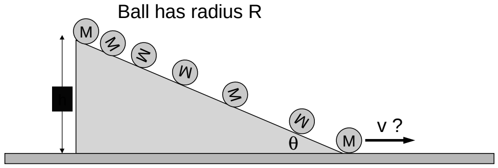
- Again consider a cylinder rolling at a constant speed.



Example : Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane ?

Ball has radius R



Lecture 14, Recap

Agenda: Chapter 10, Finish, Chapter 11, Start

- Chapter 10:
  - ❖ Moments of Inertia
  - ❖ Parallel axis theorem
  - ❖ Torque
  - ❖ Energy and Work
- Chapter 11
  - ❖ Vector Cross Products
  - ❖ Rolling Motion
  - ❖ Angular Momentum

Assignment: For Wednesday reread Chapter 11, Start Chapter 12

- WebAssign Problem Set 5 due Tuesday