## Physics 207, Lecture 15, Oct. 25

Agenda: Chapter 11, Finish, Chapter 12, Just Start

- Chapter 11:
  - ❖ Rolling Motion
  - ❖ Angular Momentum
- Chapter 12
  - ❖ Statics

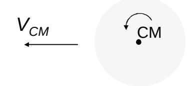
Assignment: For Monday read Chapter 12

- WebAssign Problem Set 6 due Tuesday
- Problem Set 6, Ch 10-79, Ch 11-17,23,30,35,44abdef Ch 12-4,9,21,32,35



## Rolling Motion

- Now consider a cylinder rolling at a constant speed.



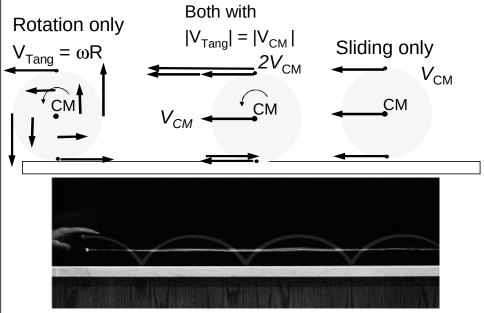

The cylinder is rotating about CM and its CM is moving at constant speed ($V_{CM}$). Thus its total kinetic energy is given by :

$$K_{TOT} = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M V_{CM}^2$$



## Motion

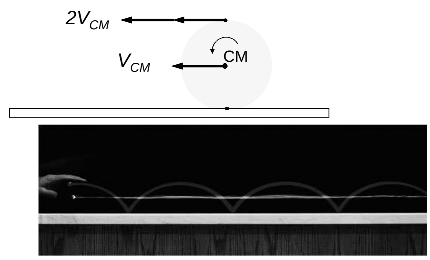
- Again consider a cylinder rolling at a constant speed.






## Rolling Motion

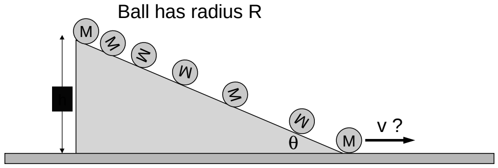
- Again consider a cylinder rolling at a constant speed.






## Example : Rolling Motion

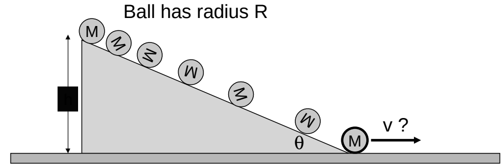
- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane ?






## Example : Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane ?
- Use Work-Energy theorem



$p = mv$

## Angular Momentum: Definitions & Derivations

- We have shown that for a system of particles

$$\vec{F}_{EXT} = \frac{d\vec{p}}{dt} \Rightarrow \text{Momentum is conserved if } F_{EXT} = 0$$

- What is the rotational equivalent of this?
- The rotational analog of force ***F*** is torque $\vec{\tau} = \vec{r} \times \vec{F}$
- Define the rotational analog of momentum ***p*** to be angular momentum, ***L*** or $\vec{L} \equiv \vec{r} \times \vec{p}$



## Recall from Chapter 9: Linear Momentum

- **Definition:** For a single particle, the momentum ***p*** is defined as:

$$\boldsymbol{p} \equiv m\boldsymbol{v}$$

(***p*** is a vector since ***v*** is a vector)

So $p_x = mv_x$ etc.

- Newton's 2nd Law:

$\boldsymbol{F} = m\boldsymbol{a}$

$$= m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \Rightarrow \vec{F} = \frac{d\vec{p}}{dt}$$

- Units of linear momentum are kg m/s.



## Linear Momentum and Angular Momentum

- **So from:**

$$\vec{p} = m\vec{v} \Rightarrow \vec{L} = I\vec{\omega}$$

- Newton's 2nd Law:

$\boldsymbol{F} = m\boldsymbol{a}$

$$= m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

- Units of angular momentum are kg m$^2$/s.



## Putting it all together

- $\vec{\tau} = \dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$

$$d\vec{L} = \vec{r} \times \vec{F}\, dt$$

$$\vec{L} = \vec{r} \times \vec{p} \Leftarrow \Delta\vec{L} = \vec{r} \times \int \vec{F}\, dt = \vec{r} \times \Delta\vec{p}$$

$$\tau_{EXT} = \frac{d\boldsymbol{L}}{dt} \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \qquad \tau_{EXT} = \boldsymbol{r} \times \boldsymbol{F}_{EXT}$$

- In the absence of external torques

Total angular momentum is conserved $\qquad \tau_{EXT} = \dfrac{d\boldsymbol{L}}{dt} = 0$

Active torque — Active angular momentum



## Conservation of angular momentum has consequences




## Angular momentum of a rigid body about a fixed axis:

- Consider a rigid distribution of point particles rotating in the *x-y* plane around the *z* axis, as shown below. The total angular momentum around the origin is the sum of the angular momentum of each particle:

(since $r_i$, $v_i$, are perpendicular)

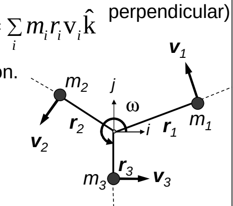
$$\boldsymbol{L}_z = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i = \sum_i m_i r_i v_i \hat{k}$$

We see that ***L*** is in the *z* direction.

Using $v_i = \omega r_i$, we get

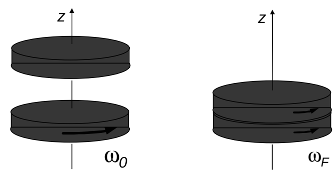
$$L_z = \sum_i m_i r_i^2 \omega \hat{k}$$

$$\vec{L} = I\vec{\omega}$$



## Example: Two Disks

- A disk of mass $M$ and radius $R$ rotates around the $z$ axis with angular velocity $\omega_0$. A second identical disk, initially not rotating, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_F$.



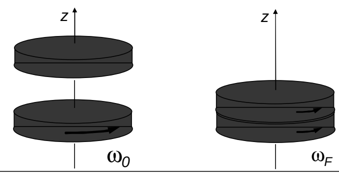



## Example: Two Disks

- A disk of mass $M$ and radius $R$ rotates around the $z$ axis with initial angular velocity $\omega_0$. A second identical disk, at rest, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_F$.

*No External Torque so $\boldsymbol{L}_z$ is constant*

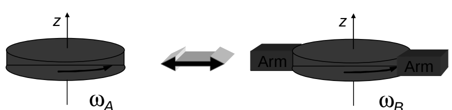
$L_i = L_f \rightarrow I\,\omega_0 + 0 = I_f\,\omega_f$






## Demonstration: Conservation of Angular Momentum

- Figure Skating :




$L_A \;=\; L_B$

$I_A \quad I_B$

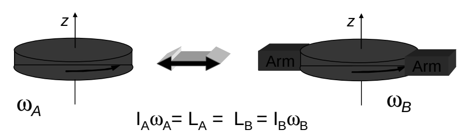
$\omega_A \quad \omega_B$

No External Torque so $\boldsymbol{L}_z$ is constant even if internal work done.



## Demonstration: Conservation of Angular Momentum

- Figure Skating :




$I_A\omega_A = L_A = L_B = I_B\omega_B$

$I_A < I_B$

$\omega_A > \omega_B$

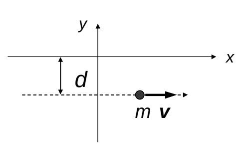
$\frac{1}{2} I_A\omega_A^2 > \frac{1}{2} I_B\,\omega_B^2$ (work needs to be done)

No External Torque so $\boldsymbol{L}_z$ is constant even if internal work done.



## Angular Momentum Conservation

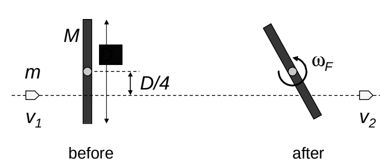
- A freely moving particle has a well defined angular momentum about any given axis.
- If no torques are acting on the particle, its angular momentum remains constant (i.e., will be conserved).
- In the example below, the direction of $\boldsymbol{L}$ is along the $z$ axis, and its magnitude is given by $L_Z = pd = mvd$.






## Example: Bullet hitting stick

- A uniform stick of mass $M$ and length $D$ is pivoted at the center. A bullet of mass $m$ is shot through the stick at a point halfway between the pivot and the end. The initial speed of the bullet is $v_1$, and the final speed is $v_2$.
  - ❖ What is the angular speed $\omega_F$ of the stick after the collision? (Ignore gravity)



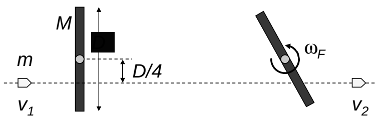
**Example: Bullet hitting stick**

- What is the angular speed $\omega_F$ of the stick after the collision? (Ignore gravity).
- Process: (1) Define system (2) Identify Conditions

(1) System: bullet and stick (No Ext. torque, L is constant)

(2) Momentum is conserved ($I_{stick} = I = MD^2/12$ )

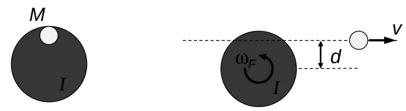
$L_{init} = L_{final}$






**Example: Throwing ball from stool**

- A student sits on a stool, initially at rest, but which is free to rotate. The moment of inertia of the student plus the stool is $I$. They throw a heavy ball of mass $M$ with speed $v$ such that its velocity vector moves a distance $d$ from the axis of rotation.
  - ❖ What is the angular speed $\omega_F$ of the student-stool system after they throw the ball ?



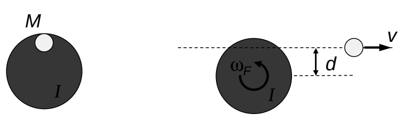



**Example: Throwing ball from stool**

- What is the angular speed $\omega_F$ of the student-stool system after they throw the ball ?
- Process: (1) Define system (2) Identify Conditions

(1) System: student, stool and ball (No Ext. torque, L is constant)

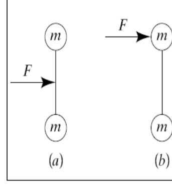
(2) Momentum is conserved






**Lecture 15, *Exercise 1***
**Concepts**

- A constant force $F$ is applied to a dumbbell for a time interval $\Delta t$, first as in case (*a*) and then as in case (*b*). Remember W=F $\Delta$x but I (impulse) = F $\Delta$t
- In which case does the dumbbell acquire the greater center-of-mass speed? (The bar is massless and rigid.)



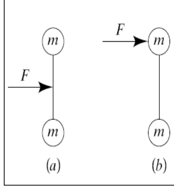

1. (*a*)
2. (*b*)
3. No difference
4. The answer depends on the rotational inertia of the dumbbell.



**Lecture 15, *Exercise 2***
**Concepts**

- A constant force $F$ is applied to a dumbbell for a time interval $\Delta t$, first as in case (*a*) and then as in case (*b*). Remember W=F $\Delta$x but I (impulse) = F $\Delta$t
- In which case does the dumbbell acquire the greater kinetic energy? (The bar is massless and rigid.)

$$K_{TOT} = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M V_{CM}^2$$



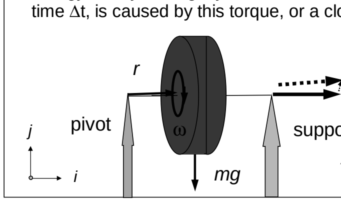

1. (*a*)
2. (*b*)
3. No difference
4. The answer depends on the rotational inertia of the dumbbell.



$\boldsymbol{\tau}_{EXT} = \frac{d\boldsymbol{L}}{dt}$ **Gyroscopic Motion:** $\boldsymbol{\tau}_{EXT} = \boldsymbol{r} \times \boldsymbol{F}_{EXT}$

- Suppose you have a spinning gyroscope in the configuration shown below:
- If the right support is removed, what will happen?
- Notice that there is a “torque” (mgr) into the display

The gyro may fall slightly but there is $\Delta$L (a vector), which in time $\Delta$t, is caused by this torque, or a clockwise rotation.




$L_x = I\,\omega_x$ (x dir)

$\Delta L = \tau\,\Delta\tau$ (-z dir)

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

**Summary of rotation:**

Comparison between Rotation and Linear Motion

| Angular | Linear |
|---|---|
| $\theta = x / R$ | $x$ |
| $\omega = v / R$ | $v$ |
| $\alpha = a / R$ | $a$ |

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**Comparison**
**Kinematics**

| Angular | Linear |
|---|---|
| $\alpha = constant$ | $a = \text{constant}$ |
| $\omega = \omega_0 + \alpha t$ | $v = v_0 + at$ |
| $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ | $x = x_0 + v_0 t + \frac{1}{2} a t^2$ |
| $\omega^2 - \omega_0^{\ 2} = 2\alpha\theta$ | $v^2 - v_0^{\ 2} = 2ax$ |
| $\omega_{AVE} = \frac{1}{2}(\omega + \omega_0)$ | $v_{AVE} = \frac{1}{2}(v + v_0)$ |

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**Comparison:** Dynamics

| Angular | Linear |
|---|---|
| $I = \Sigma_i \ m_i r_i^2$ | $m$ |
| $\tau = r \times F = \alpha I$ | $F = a m$ |
| $L = r \times \tau = I \omega$ | $p = mv$ |
| $\tau_{EXT} = \frac{d\mathbf{L}}{dt}$ | $F_{EXT} = \frac{d\mathbf{p}}{dt}$ |
| $W = \tau \Delta\theta$ | $W = \mathbf{F} \bullet \mathbf{\Delta x}$ |
| $K = \frac{1}{2} I\omega^2$ | $K = \frac{1}{2} mv^2$ |
| $\Delta K = W_{NET}$ | $\Delta K = W_{NET}$ |

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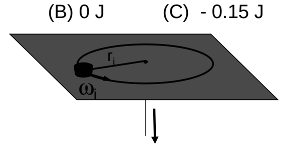
## Lecture 15, *Exercise 3*

- A mass m=0.10 kg is attached to a cord passing through a small hole in a frictionless, horizontal surface as in the Figure. The mass is initially orbiting with speed $\omega_i$ = 5 rad/s in a circle of radius $r_i$ = 0.20 m. The cord is then slowly pulled from below, and the radius decreases to r = 0.10 m. How much work is done moving the mass from $r_i$ to r ?

(A) 0.15 J (B) 0 J (C) - 0.15 J

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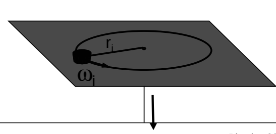
**Lecture 15, *Exercise 3***

- A mass m=0.10 kg is attached to a cord passing through a small hole in a frictionless, horizontal surface as in the Figure. The mass is initially orbiting with speed $\omega_i$ = 5 rad/s in a circle of radius $r_i$ = 0.20 m. The cord is then slowly pulled from below, and the radius decreases to r = 0.10 m. How much work is done moving the mass from $r_i$ to r ?
- Principle: No external torque so *L* is constant

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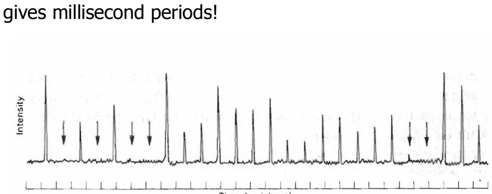
**An example: Neutron Star rotation**

Neutron star with a mass of 1.5 solar masses has a diameter of ~11 km.

Our sun rotates about once every 37 days

$\omega_f / \omega_i = I_i / I_f = r_i^2 / r_f^2 = (7\text{x}10^5 \text{ km})^2/(11 \text{ km})^2 = 4 \text{ x } 10^9$

gives millisecond periods!




period of pulsar is 1.187911164 s

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### Angular Momentum as a Fundamental Quantity

- The concept of angular momentum is also valid on a submicroscopic scale
- Angular momentum has been used in the development of modern theories of atomic, molecular and nuclear physics
- In these systems, the angular momentum has been found to be a fundamental quantity
  - ❖ Fundamental here means that it is an intrinsic property of these objects



### Fundamental Angular Momentum

- Angular momentum has discrete values
- These discrete values are multiples of a fundamental unit of angular momentum
- The fundamental unit of angular momentum is h-bar

❖ Where *h* is called Planck's constant

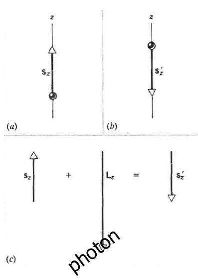
$$\hbar = \frac{h}{2\pi} = 1.054\times10^{-34}\ \frac{\text{kg}\cdot\text{m}}{\text{s}}$$

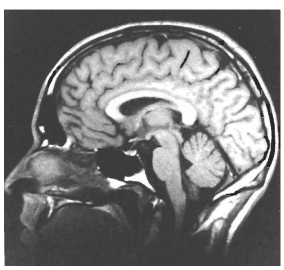
$$\Delta L = n\hbar \ (n = 1,2,3,...)$$



### Intrinsic Angular Momentum

intrinsic angular momentum of a proton is $\hbar/2$



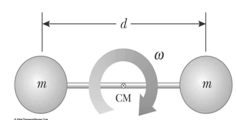





### Angular Momentum of a Molecule

- Consider the molecule as a rigid rotor, with the two atoms separated by a fixed distance
- The rotation occurs about the center of mass in the plane of the page with a speed of

$$\omega \approx \frac{\hbar}{I_{CM}}$$

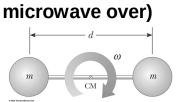




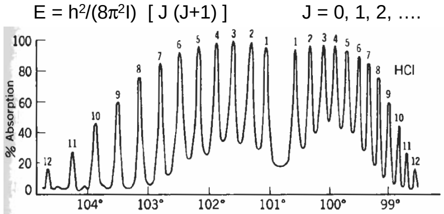

### Angular Momentum of a Molecule (It heats the water in a microwave over)

$$\Delta L = \hbar$$

$$\Delta\omega = \hbar / I_{CM}$$

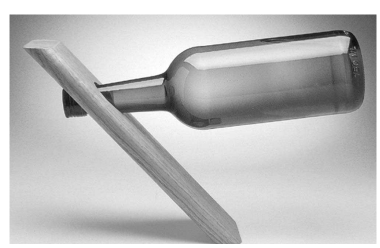



E = $h^2/(8\pi^2 I)$ [ J (J+1) ]   J = 0, 1, 2, ….





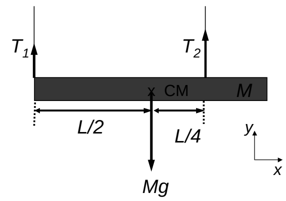

### Statics (Chapter 12): A repeat of Newton's Laws with no net force and no net torque



## Statics: Using Torque

- Now consider a plank of mass *M* suspended by two strings as shown.
- We want to find the tension in each string:

$$\Sigma \vec{F} = 0 \qquad \Sigma \vec{\tau} = 0$$






**Approach to Statics:**

- In general, we can use the two equations

$$\sum \mathbf{F} = 0 \qquad \sum \tau = 0$$

to solve any statics problems.

- When choosing axes about which to calculate torque, choose one that makes the problem easy....



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