Physics 207 – Lecture 16, Oct. 29

Agenda: Chapter 13
- Center of Mass
- Torque
- Moment of Inertia
- Rotational Energy
- Rotational Momentum

Assignment:
- Wednesday is an exam review session, Exam will be held in rooms B102 & B130 in Van Vleck at 7:15 PM
- MP Homework 7, Ch. 11, 5 problems,
  NOTE: Due Wednesday at 4 PM
- MP Homework 7A, Ch. 13, 5 problems, available soon

Chap. 13: Rotational Dynamics
- Up until now rotation has been only in terms of circular motion with $a_c = \frac{v^2}{R}$ and $|a_T| = \frac{dv}{dt}$
- Rotation is common in the world around us.
- Many ideas developed for translational motion are transferable.

Conservation of angular momentum has consequences
How does one describe rotation (magnitude and direction)?

Rotational Dynamics: A child’s toy, a physics playground or a student’s nightmare
- A merry-go-round is spinning and we run and jump on it. What does it do?
- We are standing on the rim and our “friends” spin it faster. What happens to us?
- We are standing on the rim a walk towards the center. Does anything change?

Rotational Variables
- Rotation about a fixed axis:
  - Consider a disk rotating about an axis through its center
  - How do we describe the motion:
    $\omega = \frac{d\theta}{dt} = \frac{2\pi}{T}$ (rad/s) = $v_{\text{tangential}}R$
    (Analogous to the linear case $v = \frac{dx}{dt}$)

Rotational Variables...
- Recall: At a point a distance $R$ away from the axis of rotation, the tangential motion:
  - $x = \theta R$
  - $v = \omega R$
  - $a = \alpha R$

$\alpha = \text{constant}$ (angular acceleration in rad/s²)
$\omega = \omega_0 + \alpha t$ (angular velocity in rad/s)
$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$ (angular position in rad)
Physics 207 – Lecture 12

Summary (with comparison to 1-D kinematics)

<table>
<thead>
<tr>
<th>Angular</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \text{constant}$</td>
<td>$a = \text{constant}$</td>
</tr>
<tr>
<td>$\omega = \omega_0 + \alpha t$</td>
<td>$v = v_0 + at$</td>
</tr>
<tr>
<td>$\theta = \theta_0 + \frac{1}{2} \alpha t^2$</td>
<td>$x = x_0 + v_0 t + \frac{1}{2} at^2$</td>
</tr>
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And for a point at a distance $R$ from the rotation axis:

$$x = R \theta$$

$$v = R \omega$$

$$a = R \alpha$$

Lecture 15, Exercise 5

Rotational Definitions

- A goofy friend sees a disk spinning and says “Ooh, look! There’s a wheel with a negative $\omega$ and with antiparallel $\omega$ and $\alpha$!”

- Which of the following is a true statement about the wheel?
  
  (A) The wheel is spinning counter-clockwise and slowing down.
  
  (B) The wheel is spinning counter-clockwise and speeding up.
  
  (C) The wheel is spinning clockwise and slowing down.
  
  (D) The wheel is spinning clockwise and speeding up.

Example: Wheel And Rope

- A wheel with radius $r = 0.4 \text{ m}$ rotates freely about a fixed axle. There is a rope wound around the wheel.

  Starting from rest at $t = 0$, the rope is pulled such that it has a constant acceleration $a = 4 \text{ m/s}^2$. How many revolutions has the wheel made after 10 seconds?

  (One revolution $= 2\pi$ radians)

  $$a = \alpha r$$

  $$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

  $$R = \frac{(\theta - \theta_0)}{2\pi} = 0 + \frac{1}{2} \left(\frac{a}{r}\right) t^2 / 2\pi$$

  $$R = \frac{0.5 \times 10 \times 100}{6.28}$$

System of Particles (Distributed Mass):

- Until now, we have considered the behavior of very simple systems (one or two masses).

- But real objects have distributed mass!

- For example, consider a simple rotating disk and 2 equal mass $m$ plugs at distances $r$ and $2r$.

  - Compare the velocities and kinetic energies at these two points.

  $$K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2$$

  $$K = \frac{1}{2} m (2v)^2 = \frac{1}{2} m (\omega 2r)^2$$

  - The rotation axis matters too!
System of Particles: Center of Mass

- If an object is not held then it rotates about the center of mass.
- Center of mass: Where the system is balanced!
  - Building a mobile is an exercise in finding centers of mass.

If an object is not held then it rotates about the center of mass.

Center of mass: Where the system is balanced!

Building a mobile is an exercise in finding centers of mass.

\[ \mathbf{R}_{CM} = \frac{\sum m_i \mathbf{r}_i}{M} = \mathbf{X}_{CM} \mathbf{i} + \mathbf{Y}_{CM} \mathbf{j} + \mathbf{Z}_{CM} \mathbf{k} \]

For a collection of \( N \) individual pointlike particles whose masses and positions we know:

\[ \mathbf{R}_{CM} = \frac{\sum m_i \mathbf{r}_i}{M} \]

(In this case, \( N = 2 \))

Sample calculation:

Consider the following mass distribution:

\[ \mathbf{R}_{CM} = \frac{\sum m_i \mathbf{r}_i}{M} = \mathbf{X}_{CM} \mathbf{i} + \mathbf{Y}_{CM} \mathbf{j} + \mathbf{Z}_{CM} \mathbf{k} \]

\[ X_{CM} = (mx_0 + 2mx_2 + mx_4)/4m \text{ meters} \]

\[ Y_{CM} = (mx_0 + 2mx_2 + mx_0)/4m \text{ meters} \]

\[ X_{CM} = 12 \text{ meters} \]

\[ Y_{CM} = 6 \text{ meters} \]

Rotational Dynamics: What makes it spin?

A force applied at a distance from the rotation axis

\[ \tau_{TOT} = |r| |\mathbf{F}_{\text{Tang}}| = |r| |\mathbf{F}| \sin \phi \]

- Torque is the rotational equivalent of force
  - Torque has units of $\text{kg m}^2/\text{s}^2 = (\text{kg m/s}^2) \text{ m} = \text{N m}$

- A constant torque gives constant angular acceleration iff the mass distribution and the axis of rotation remain constant.

System of Particles: Center of Mass

- How do we describe the "position" of a system made up of many parts?
- Define the Center of Mass (average position):

For a continuous solid, convert sums to an integral.

\[ \mathbf{R}_{CM} = \int \mathbf{r} \, dm \]

where \( dm \) is an infinitesimal mass element.

Rotational Dynamics: What makes it spin?

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Lecture 16, Exercise 1

Torque

- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.
- Remember torque requires \( \mathbf{F}, r \) and \( \sin \theta \) or the tangential force component times perpendicular distance

A. Case 1
B. Case 2
C. Same
Lecture 16, Exercise 1

Torque

- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.
- Remember torque requires $F$, $r$, and $\sin \phi$ or the tangential force component times perpendicular distance

(A) case 1
(B) case 2
(C) same

Rotational Dynamics: What makes it spin?

A force applied at a distance from the rotation axis

$$\tau_{TOT} = |r| |F_{Tang}| \equiv |r| |F| \sin \phi$$

- Torque is the rotational equivalent of force
- Torque has units of kg m²/s² = (kg m/s²) m = N m

$$\tau_{TOT} = r F_{Tang} = r m a$$
$$= r m r \alpha$$
$$= m r^2 \alpha$$

For every little part of the wheel

Calculating Moment of Inertia

$$I \equiv \sum_{i=1}^{N} m_i r_i^2$$
where $r$ is the distance from the mass to the axis of rotation.

Example: Calculate the moment of inertia of four point masses $(m)$ on the corners of a square whose sides have length $L$, about a perpendicular axis through the center of the square:

$$I = \frac{2mL^2}{4}$$

Lecture 16, Home Exercise

Moment of Inertia

- A triangular shape is made from identical balls and identical rigid, massless rods as shown. The moment of inertia about the $a$, $b$, and $c$ axes is $I_a$, $I_b$, and $I_c$ respectively.
- Which of the following is correct:

(A) $I_a > I_b > I_c$
(B) $I_b > I_c > I_a$
(C) $I_a > I_c > I_b$
Lecture 16, Home Exercise
Moment of Inertia

- \( I_a = 2 \, \text{m} \, (2L)^2 \)
- \( I_b = 3 \, \text{m} \, L^2 \)
- \( I_c = m \, (2L)^2 \)
- Which of the following is correct:
  - (A) \( I_a > I_b > I_c \)
  - (B) \( I_a > I_c > I_b \)
  - (C) \( I_b > I_a > I_c \)

Calculating Moment of Inertia...

- For a discrete collection of point masses we found:
  \[
  I = \sum_{i=1}^{N} m_i r_i^2
  \]
- For a continuous solid object we have to add up the \( m^2 \) contribution for every infinitesimal mass element \( dm \).
- An integral is required to find \( I \):
  \[
  I = \int r^2 \, dm
  \]

Moments of Inertia

- Some examples of \( I \) for solid objects:
  - Solid disk or cylinder of mass \( M \) and radius \( R \), about perpendicular axis through its center.
    \[
    I = \frac{1}{2} M R^2
    \]

Moments of Inertia...

- Some examples of \( I \) for solid objects:
  - Solid sphere of mass \( M \) and radius \( R \), about an axis through its center.
    \[
    I = \frac{2}{5} M R^2
    \]
  - Thin spherical shell of mass \( M \) and radius \( R \), about an axis through its center.
    Use the table...
    See Table 13.3, Moments of Inertia

Moments of Inertia

- Some examples of \( I \) for solid objects:
  - Thin hoop (or cylinder) of mass \( M \) and radius \( R \), about an axis through its center, perpendicular to the plane of the hoop is just \( MR^2 \)
  - Thin hoop of mass \( M \) and radius \( R \), about an axis through a diameter.

Rotation & Kinetic Energy

- Consider the simple rotating system shown below.
  (Assume the masses are attached to the rotation axis by massless rigid rods).
- The kinetic energy of this system will be the sum of the kinetic energy of each piece:
  \[
  K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2
  \]
- \( K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \frac{1}{2} m_4 v_4^2 \)
Lecture 16, Rotation & Kinetic Energy

Rotation & Kinetic Energy

- Notice that $v_1 = \alpha r_1$, $v_2 = \alpha r_2$, $v_3 = \alpha r_3$, $v_4 = \alpha r_4$
- So we can rewrite the summation:

$$K = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i \omega^2 r_i^2 = \frac{1}{2} \sum m_i r_i^2 \omega^2$$

- We recognize the quantity, moment of inertia or $I$, and write:

$$K = \frac{1}{2} I \omega^2$$

Moment of Inertia and Rotational Energy

- So $K = \frac{1}{2} I \omega^2$ where $I = \sum m_i r_i^2$
- Notice that the moment of inertia $I$ depends on the distribution of mass in the system.
- The further the mass is from the rotation axis, the bigger the moment of inertia.
- For a given object, the moment of inertia depends on where we choose the rotation axis (unlike the center of mass).
- In rotational dynamics, the moment of inertia $I$ appears in the same way that mass $m$ does in linear dynamics!

Lecture 16, Exercise 2

Rotational Kinetic Energy

- We have two balls of the same mass. Ball 1 is attached to a 0.1 m long rope. It spins around at 2 revolutions per second. Ball 2 is on a 0.2 m long rope. It spins around at 2 revolutions per second.

$$K = \frac{1}{2} I \omega^2$$

- What is the ratio of the kinetic energy of Ball 2 to that of Ball 1?

(A) 1/4 (B) 1/2 (C) 1 (D) 2 (E) 4

Work (in rotational motion)

- Consider the work done by a force $F$ acting on an object constrained to move around a fixed axis. For an infinitesimal angular displacement $d\theta$ where $d\theta = \hat{R} d\theta$

$$dW = F \text{\,tan} \cdot \,dr$$

$$dW = (F \text{\,tan} \theta) \, d\theta$$

- We can integrate this to find: $W = \tau \theta = \tau \theta (\hat{r} \times \hat{r})$
- Analogue of $W = F \cdot \Delta r$
- $W$ will be negative if $\tau$ and $\theta$ have opposite sign!
Recall the Work Kinetic-Energy Theorem: \( \Delta K = W_{NET} \)

This is true in general, and hence applies to rotational motion as well as linear motion.

So for an object that rotates about a fixed axis:

\[
\Delta K = \frac{1}{2} I (\omega_f^2 - \omega_i^2) = W_{NET}
\]

Lecture 16, Home exercise

Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same linear distance.

Disk 1 has a bigger radius, but both are identical material (i.e. their density \( \rho = M/V \) is the same). Both disks rotate freely around axes through their centers, and start at rest.

Which disk has the biggest angular velocity after the pull?

\[
W = \tau \theta = F d = \frac{1}{2} I \omega^2
\]

(A) Disk 1
(B) Disk 2
(C) Same

Example: Rotating Rod

A uniform rod of length \( L=0.5 \text{ m} \) and mass \( m=1 \text{ kg} \) is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is

1. For forces you need to locate the Center of Mass
   CM is at \( L/2 \) (halfway) and put in the Force on a FBD
2. The hinge changes everything!

\[
F_x = 0 \quad \text{occurs only at the hinge}
\]

but \( \tau_z = I \alpha_z = r F \sin 90^\circ \)

at the center of mass and

\[
\alpha_z = (I_{CM} + m(L/2)^2) / (L/2) mg
\]

and solve for \( \alpha_z \)

Example: Rotating Rod

A uniform rod of length \( L=0.5 \text{ m} \) and mass \( m=1 \text{ kg} \) is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is

1. For forces you need to locate the Center of Mass
   CM is at \( L/2 \) (halfway) and put in the Force on a FBD
2. The hinge changes everything!

\[
a = \alpha L
\]
Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is (A) its angular speed when it reaches the lowest point?
  1. For forces you need to locate the Center of Mass (CM) is at L/2 (halfway) and use the Work-Energy Theorem.
  2. The hinge changes everything!

\[
W = mg \cdot \frac{L}{2} = \frac{1}{2} (I_{CM} + m \cdot \left(\frac{L}{2}\right)^2) \omega^2
\]

and solve for \( \omega \).

---

Connection with CM motion

- If an object of mass \( M \) is moving linearly at velocity \( V_{CM} \) without rotating then its kinetic energy is

\[
K_T = \frac{1}{2} M V_{CM}^2
\]

- If an object of moment of inertia \( I_{CM} \) is rotating in place about its center of mass at angular velocity \( \omega \) then its kinetic energy is

\[
K_T = \frac{1}{2} I_{CM} \omega^2
\]

- What if the object is both moving linearly and rotating?

\[
K_T = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M V_{CM}^2
\]

---

Connection with CM motion...

- So for a solid object which rotates about its center of mass and whose CM is moving:

\[
K_{TOT} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M V_{CM}^2
\]

---

Example: Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane?

\[
Mgh = \frac{1}{2} M v^2 + \frac{1}{2} I_{CM} \omega^2
\]

---

Example : Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane?

- Use Work-Energy theorem

\[
Mgh = \frac{1}{2} M v^2 + \frac{1}{2} I_{CM} \omega^2
\]

\[
Mgh = \frac{1}{2} M v^2 + \frac{1}{2} (\frac{1}{2} M R^2) (\frac{v}{R})^2 = \frac{3}{4} M v^2
\]

\[
v = 2 \left(\frac{gh}{3}\right)^{1/3}
\]
Rolling Motion

- Now consider a cylinder rolling at a constant speed.

The cylinder is rotating about CM and its CM is moving at constant speed \( V_{CM} \). Thus its total kinetic energy is given by:

\[
K_{TOT} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M V_{CM}^2
\]

Motion

- Again consider a cylinder rolling at a constant speed.

Angular Momentum:

- We have shown that for a system of particles, momentum \( p = m \dot{v} \) is conserved if \( \dot{p} = 0 \).

- What is the rotational equivalent of this? Angular momentum \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) is conserved if \( \dot{\mathbf{L}} = 0 \).

Example: Two Disks

- A disk of mass \( M \) and radius \( R \) rotates around the z axis with initial angular velocity \( \omega_0 \). A second identical disk, initially at rest, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity \( \omega_f \).

No External Torque so \( \mathbf{L} \) is constant

\[
L_z = I z = I \omega_0 \rightarrow I \omega_0 z = I \omega_f z \rightarrow \frac{1}{2} m R^2 \omega_0 = \frac{1}{2} 2mR^2 \omega_f
\]

Lecture 16, Oct. 29

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An example: Neutron Star rotation

A neutron star with a mass of 1.5 solar masses has a diameter of ~11 km.

Our sun rotates about once every 37 days.

\[
\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f} = \frac{r_i^2}{r_f^2} = \left(\frac{7 \times 10^5 \text{ km}}{11 \text{ km}}\right)^2 = 4 \times 10^5
\]

gives millisecond periods!

\[
\text{period of pulsar is } 1.187911164 \text{ s}
\]

Angular Momentum as a Fundamental Quantity

- The concept of angular momentum is also valid on a submicroscopic scale.
- Angular momentum has been used in the development of modern theories of atomic, molecular, and nuclear physics.
- In these systems, the angular momentum has been found to be a fundamental quantity.
  - Fundamental here means that it is an intrinsic property of these objects.

Fundamental Angular Momentum

- Angular momentum has discrete values.
- These discrete values are multiples of a fundamental unit of angular momentum.
- The fundamental unit of angular momentum is h-bar.
  - Where \( h \) is called Planck's constant.
  - \( h = \frac{\hbar}{2\pi} = 1.054 \times 10^{-34} \text{ kg} \cdot \text{m} / \text{s} \)
  - \( \Delta L = n\hbar \) (\( n = 1, 2, 3, \ldots \))

Intrinsic Angular Momentum

- The intrinsic angular momentum of a proton is \( \hbar / 2 \).

Angular Momentum of a Molecule

- Consider the molecule as a rigid rotor, with the two atoms separated by a fixed distance.
- The rotation occurs about the center of mass in the plane of the page with a speed of
  \[
  \omega \approx \frac{\hbar}{I_{CM}}
  \]

Angular Momentum of a Molecule (It heats the water in a microwave over)

\[
\Delta L = \hbar
\]
\[
\Delta \omega = \hbar / I_{CM}
\]
\[
E = \hbar^2/(8\pi^2 J (J+1)) \quad J = 0, 1, 2, \ldots
\]