## Physics 207, Lecture 16, Oct. 29

Agenda: Chapter 13

- Center of Mass
- Torque
- Moment of Inertia
- Rotational Energy
- Rotational Momentum

Assignment:

- Wednesday is an exam review session, Exam will be held in rooms B102 & B130 in Van Vleck at 7:15 PM
- MP Homework 7, Ch. 11, 5 problems,
  NOTE: Due Wednesday at 4 PM
- MP Homework 7A, Ch. 13, 5 problems, available soon

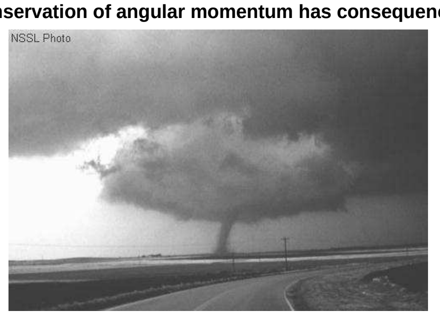


## Chap. 13: Rotational Dynamics

- Up until now rotation has been only in terms of circular motion with $a_c = v^2 / R$ and $| a_T | = d| v | / dt$
- Rotation is common in the world around us.
- Many ideas developed for translational motion are transferable.



### Conservation of angular momentum has consequences




How does one describe rotation (magnitude and direction)?



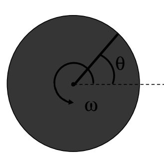
## Rotational Dynamics: A child's toy, a physics playground or a student's nightmare

- A merry-go-round is spinning and we run and jump on it. What does it do?
- We are standing on the rim and our "friends" spin it faster. What happens to us?
- We are standing on the rim a walk towards the center. Does anything change?



## Rotational Variables

- Rotation about a fixed axis:
  - ❖ Consider a disk rotating about an axis through its center:]




- How do we describe the motion:

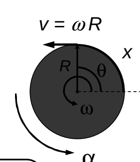
$$\omega = \frac{d\theta}{dt} = \frac{2\pi}{T}\,(\text{rad/s}) = v_{\text{Tangential}}\text{R}$$

(Analogous to the linear case $v = \frac{dx}{dt}$ )



## Rotational Variables...

- Recall: At a point a distance *R* away from the axis of rotation, the tangential motion:
  - ❖ $x = \theta R$
  - ❖ $v = \omega R$
  - ❖ $a = \alpha R$




$$\boxed{\begin{array}{ll} \alpha = \text{constant} & (\text{angular accelation in rad/s}^2) \\ \omega = \omega_0 + \alpha t & (\text{angular velocity in rad/s}) \\ \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 & (\text{angular position in rad}) \end{array}}$$

### Summary
### (with comparison to 1-D kinematics)

| Angular | Linear |
|---|---|
| $\alpha = \text{constant}$ | $a = \text{constant}$ |
| $\omega = \omega_0 + \alpha t$ | $v = v_0 + at$ |
| $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ | $x = x_0 + v_0 t + \frac{1}{2} a t^2$ |

And for a point at a distance *R* from the rotation axis:

$x = R\,\theta$ $\quad v = \omega R$ $\quad a = \alpha R$



### Lecture 15, *Exercise 5*
### Rotational Definitions

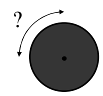
- A goofy friend sees a disk spinning and says "Ooh, look! There's a wheel with a negative $\omega$ and with antiparallel $\omega$ and $\alpha$!"
- Which of the following is a true statement about the wheel?

(A) The wheel is spinning counter-clockwise and slowing down.

(B) The wheel is spinning counter-clockwise and speeding up.

**(C) The wheel is spinning clockwise and slowing down.**

(D) The wheel is spinning clockwise and speeding up





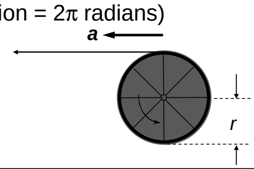

### Example: Wheel And Rope

- A wheel with radius *r = 0.4* m rotates freely about a fixed axle. There is a rope wound around the wheel.

  Starting from rest at *t = 0*, the rope is pulled such that it has a constant acceleration *a = 4m/s²*.

  How many revolutions has the wheel made after *10* seconds?

  (One revolution = $2\pi$ radians)





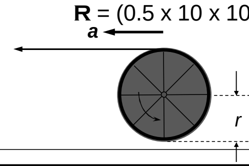

### Example: Wheel And Rope

- A wheel with radius *r = 0.4* m rotates freely about a fixed axle. There is a rope wound around the wheel. Starting from rest at *t = 0*, the rope is pulled such that it has a constant acceleration *a* = 4 m/s². How many revolutions has the wheel made after 10 seconds? (One revolution = $2\pi$ radians)
- Revolutions = **R** = $(\theta - \theta_0) / 2\pi$ and $a = \alpha r$

  $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \rightarrow$

  **R** = $(\theta - \theta_0) / 2\pi = 0 + \frac{1}{2}(a/r)\, t^2 / 2\pi$

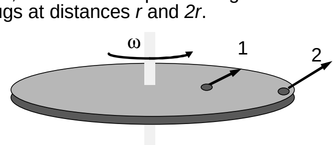
  **R** = (0.5 x 10 x 100) / 6.28






### System of Particles (Distributed Mass):

- Until now, we have considered the behavior of very simple systems (one or two masses).
- But real objects have distributed mass !
- For example, consider a simple rotating disk and 2 equal mass *m* plugs at distances *r* and *2r*.



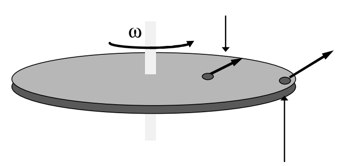

- Compare the velocities and kinetic energies at these two points.



### System of Particles (Distributed Mass):

- An extended solid object (like a disk) can be thought of as a collection of parts.
- The motion of each little part depends on where it is in the object!

1 $K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2$

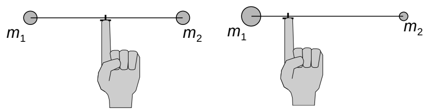



- The rotation axis matters too!

2 $K = \frac{1}{2} m (2v)^2 = \frac{1}{2} m (\omega 2r)^2$

## System of Particles: Center of Mass

- If an object is not held then it rotates about the center of mass.
- Center of mass: Where the system is balanced !
  - ❖ Building a mobile is an exercise in finding centers of mass.



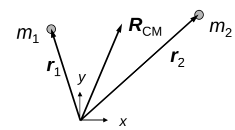

mobile



## System of Particles: Center of Mass

- How do we describe the "position" of a system made up of many parts ?
- Define the **Center of Mass** (average position):
  - ❖For a collection of *N* individual pointlike particles whose masses and positions we know:

$$\vec{\boldsymbol{R}}_{\mathrm{CM}} \equiv \frac{\sum_{i=1}^{N} m_i \vec{\boldsymbol{r}}_i}{M}$$




(In this case, $N = 2$)



## Sample calculation:

- Consider the following mass distribution:

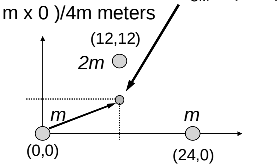
$$\vec{\boldsymbol{R}}_{\mathrm{CM}} = \frac{\sum_{i=1}^{N} m_i \vec{\boldsymbol{r}}_i}{M} = X_{\mathrm{CM}}\,\hat{\mathrm{i}} + Y_{\mathrm{CM}}\,\hat{\mathrm{j}} + Z_{\mathrm{CM}}\,\hat{\mathrm{k}}$$

$X_{\mathrm{CM}}$ = ($m$ x 0 + 2m x 12 + m x 24 )/4m meters

$Y_{\mathrm{CM}}$ = ($m$ x 0 + 2m x 12 + m x 0 )/4m meters

$X_{\mathrm{CM}}$ = 12 meters

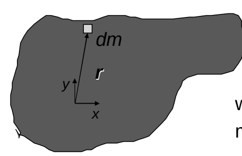
$Y_{\mathrm{CM}}$ = 6 meters






## System of Particles: Center of Mass

- For a continuous solid, convert sums to an integral.




$$\vec{\boldsymbol{R}}_{\mathrm{CM}} = \frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \vec{r}\, dm}{M}$$

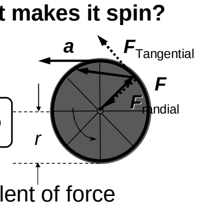
where *dm* is an infinitesimal mass element.



## Rotational Dynamics: What makes it spin?

A force applied at a distance from the rotation axis

$$\boxed{\tau_{\mathrm{TOT}} = |\mathrm{r}|\ |\mathrm{F}_{\mathrm{Tang}}| \equiv |\mathrm{r}|\ |\mathrm{F}|\ \sin\phi}$$




- Torque is the rotational equivalent of force

  Torque has units of kg $m^2/s^2$ = (kg $m/s^2$) m = N m

- A <u>constant torque</u> gives <u>constant angular acceleration</u> iff the mass distribution and the axis of rotation remain constant.

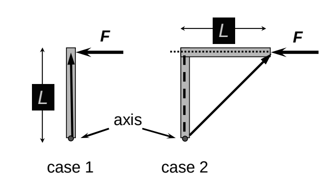


## Lecture 16, *Exercise 1*
## Torque

- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.
- Remember torque requires ***F***, ***r*** and sin θ

or the tangential force component times perpendicular distance

- A. Case 1
- B. Case 2
- C. Same



## Lecture 16, *Exercise 1*
## Torque

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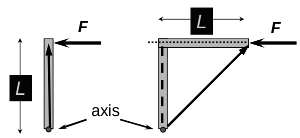
(A) case 1

(B) case 2

**(C) same**



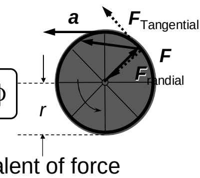

case 1 case 2



## Rotational Dynamics: What makes it spin?

A force applied at a distance from the rotation axis

$$\boxed{\tau_{TOT} = |r|\,|F_{Tang}| \equiv |r|\,|F| \sin \phi}$$




- Torque is the rotational equivalent of force

Torque has units of kg m2/s2 = (kg m/s2) m = N m

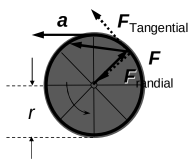
$$\tau_{TOT} = r\ F_{Tang} = r\,m\,a$$
$$= r\,m\,r\,\alpha$$
$$= m\,r^2\ \alpha$$

For every little part of the wheel



## $\tau_{TOT} = m\,r^2\,\alpha$ and inertia

The further a mass is away from this axis the greater the inertia (resistance) to rotation




- This is the rotational version of $F_{TOT} = ma$
- Moment of inertia, $I \equiv \mathbf{m\,r^2}$, (here $I$ is just a point on the wheel) is the rotational equivalent of mass.
- If $I$ is big, more torque is required to achieve a given angular acceleration.

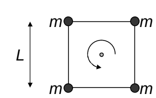


## Calculating Moment of Inertia

$$I \equiv \sum_{i=1}^{N} m_i r_i^{\,2}$$

where *r* is the distance from the mass to the axis of rotation.

Example: Calculate the moment of inertia of four point masses (*m*) on the corners of a square whose sides have length *L*, about a perpendicular axis through the center of the square:

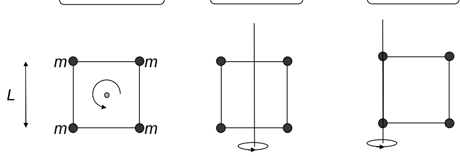





## Calculating Moment of Inertia...

- For a single object, I **depends** on the rotation axis!
- Example: $I_1 = 4\ m\ R^2 = 4\ m\ (2^{1/2}\ L\ /\ 2)^2$

$\boxed{I_1 = 2mL^2}$ $\boxed{I_2 = mL^2}$ $\boxed{I = 2mL^2}$






## Lecture 16, *Home Exercise*
## Moment of Inertia

- A triangular shape is made from identical balls and identical rigid, massless rods as shown. The moment of inertia about the *a*, *b*, and *c* axes is $I_a$, $I_b$, and $I_c$ respectively.

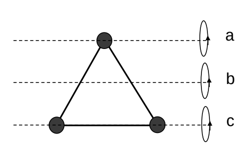
$$\boxed{I = \sum_i m_i r_i^{\,2}}$$

❖ Which of the following is correct:

(A) $I_a > I_b > I_c$

(B) $I_a > I_c > I_b$

(C) $I_b > I_a > I_c$



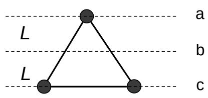
## Lecture 16, *Home Exercise* Moment of Inertia

- $I_a = 2\,m\,(2L)^2 \qquad I_b = 3\,m\,L^2 \qquad I_c = m\,(2L)^2$
- Which of the following is correct:

(A) $I_a > I_b > I_c$

**(B) $I_a > I_c > I_b$**

(C) $I_b > I_a > I_c$

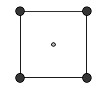





## Calculating Moment of Inertia...

- For a discrete collection of point masses we found:

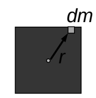
$$I = \sum_{i=1}^{N} m_i r_i^2$$



- For a continuous solid object we have to add up the $mr^2$ contribution for every infinitesimal mass element $dm$.

❖ An integral is required to find I :

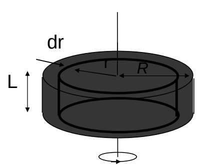
$$I = \int r^2 dm$$






## Moments of Inertia

- Some examples of I for solid objects:



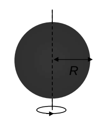

Solid disk or cylinder of mass *M* and radius *R*, about perpendicular axis through its center.

$I = ½\, M R^2$



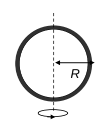
## Moments of Inertia...

- Some examples of *I* for solid objects:




Solid sphere of mass *M* and radius *R*, about an axis through its center.

I = 2/5 M R2




Thin spherical shell of mass *M* and radius *R*, about an axis through its center.

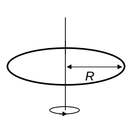
Use the table…

See Table 13.3, Moments of Inertia

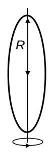


## Moments of Inertia

- Some examples of I for solid objects:




Thin hoop (or cylinder) of mass *M* and radius *R*, about an axis through it center, perpendicular to the plane of the hoop is just $MR^2$




Thin hoop of mass *M* and radius *R*, about an axis through a diameter.

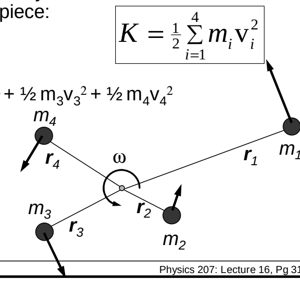


## Rotation & Kinetic Energy

- Consider the simple rotating system shown below. (Assume the masses are attached to the rotation axis by massless rigid rods).
- The kinetic energy of this system will be the sum of the kinetic energy of each piece:

$$\boxed{K = \tfrac{1}{2}\sum_{i=1}^{4} m_i v_i^2}$$

- $K = ½\, m_1 v_1^2 + ½\, m_2 v_2^2 + ½\, m_3 v_3^2 + ½\, m_4 v_4^2$



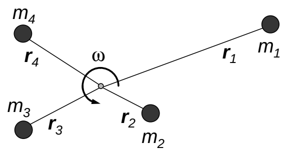
### Rotation & Kinetic Energy

- Notice that $v_1 = \omega r_1$, $v_2 = \omega r_2$, $v_3 = \omega r_3$, $v_4 = \omega r_4$
- So we can rewrite the summation:

$$\boxed{K = \tfrac{1}{2}\sum_{i=1}^{4} m_i v_i^2 = \tfrac{1}{2}\sum_{i=1}^{4} m_i \omega^2 r_i^2 = \tfrac{1}{2}\left[\sum_{i=1}^{4} m_i r_i^2\right]\omega^2}$$

- We recognize the quantity, moment of inertia or I, and write:

$$\boxed{K = \tfrac{1}{2} I \omega^2}$$

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### Lecture 16, *Exercise 2*
### Rotational Kinetic Energy

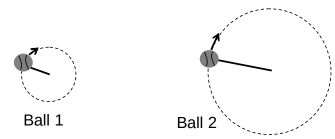
- We have two balls of the same mass. Ball 1 is attached to a 0.1 m long rope. It spins around at 2 revolutions per second. Ball 2 is on a 0.2 m long rope. It spins around at 2 revolutions per second.

$$K = \tfrac{1}{2} I \omega^2$$

- What is the ratio of the kinetic energy of Ball 2 to that of Ball 1 ?

$$I = \sum_i m_i r_i^2$$

A. ¼
B. ½
C. 1
D. 2
E. 4

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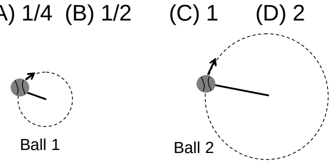
### Lecture 16, *Exercise 2*
### Rotational Kinetic Energy

- $K_2/K_1 = ½\, m\, \omega r_2^2 / ½\, m\, \omega r_1^2 = 0.2^2 / 0.1^2 = 4$
- What is the ratio of the kinetic energy of Ball 2 to that of Ball 1 ?

(A) 1/4 (B) 1/2 (C) 1 (D) 2 **(E) 4**

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### Rotation & Kinetic Energy...

- The kinetic energy of a rotating system looks similar to that of a point particle:

| Point Particle | Rotating System |
|---|---|
| $K = \tfrac{1}{2} m v^2$<br>*v* is "linear" velocity<br>*m* is the mass. | $K = \tfrac{1}{2} I \omega^2$<br>ω is angular velocity<br>*I* is the moment of inertia about the rotation axis.<br>$I = \sum_i m_i r_i^2$ |

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### Moment of Inertia and Rotational Energy

- So $K = \tfrac{1}{2} I \omega^2$ where $I = \sum_i m_i r_i^2$
- Notice that the moment of inertia I depends on the distribution of mass in the system.
  - ❖ The further the mass is from the rotation axis, the bigger the moment of inertia.
- For a given object, the moment of inertia depends on where we choose the rotation axis (unlike the center of mass).
- In rotational dynamics, the moment of inertia I appears in the same way that mass *m* does in linear dynamics !

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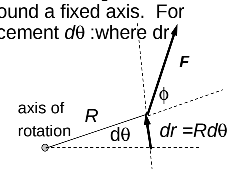
### Work (in rotational motion)

- Consider the work done by a force ***F*** acting on an object constrained to move around a fixed axis. For an infinitesimal angular displacement $d\theta$ :where dr =R $d\theta$

$dW = F_{Tangential}\, dr$

$dW = (F_{Tangential}\, R)\, d\theta$




$dW = \tau\, d\theta$ (and with a constant torque)

- We can integrate this to find: $\boxed{W = \tau\, \theta} = (\tau_f - \tau_i)\, \theta$
- Analogue of $W = \boldsymbol{F} \bullet \Delta\boldsymbol{r}$
- $W$ will be negative if $\tau$ and $\theta$ have opposite sign !

Physics 207: Lecture 16, Pg 37

### Work & Kinetic Energy:

- Recall the Work Kinetic-Energy Theorem: $\Delta K = W_{NET}$
- This is true in general, and hence applies to rotational motion as well as linear motion.
- So for an object that rotates about a fixed axis:

$$\boxed{\Delta K = \tfrac{1}{2} I \left(\omega_f^2 - \omega_i^2\right) = W_{NET}}$$



### Lecture 16, *Home exercise* Work & Energy

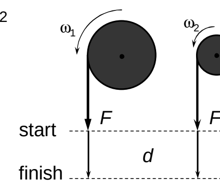
- Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same linear distance.
  Disk 1 has a bigger radius, but both are identical material (i.e. their density ρ = M/V is the same). Both disks rotate freely around axes though their centers, and start at rest.
  - ❖ Which disk has the biggest angular velocity after the pull?

$W = \tau\, \theta = F\, d = ½\, I\, \omega^2$

(A) Disk 1

(B) Disk 2

(C) Same






### Lecture 16, *Home exercise* Work & Energy

- Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same linear distance.
  Disk 1 has a bigger radius, but both are identical material (i.e. their density ρ = M/V is the same). Both disks rotate freely around axes though their centers, and start at rest.
  - ❖ Which disk has the biggest angular velocity after the pull?

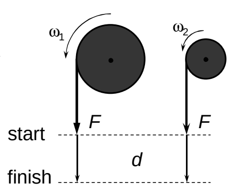
$W = F\, d = ½\, I_1\, \omega_1^2 = ½\, I_2\, \omega_2^2$

$\omega_1 = (I_2 / I_1)^{½}\ \ \omega_2$ and $I_2 < I_1$

(A) Disk 1

**(B) Disk 2**

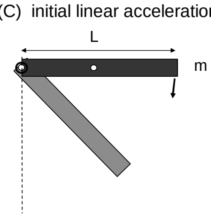
(C) Same






### Example: Rotating Rod

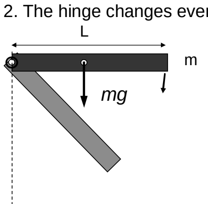
- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless pin passing through one end as in the Figure. The rod is released from rest in the horizontal position. What is
  (A) its angular speed when it reaches the lowest point ?
  (B) its initial angular acceleration ?
  (C) initial linear acceleration of its free end ?






### Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is
  (B) its initial angular acceleration ?
  1. For forces you need to locate the Center of Mass
  CM is at L/2 ( halfway ) and put in the Force on a FBD
  2. The hinge changes everything!




$\Sigma\, \boldsymbol{F} = 0$ occurs only at the hinge

but $\tau_z = I\, \alpha_z = r\, F \sin 90°$

at the center of mass and

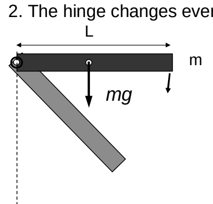
$(I_{CM} + m(L/2)^2)\ \alpha_z = (L/2)\, mg$

and solve for $\alpha_z$



### Example: Rotating Rod

- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is
  (C) initial linear acceleration of its free end ?
  1. For forces you need to locate the Center of Mass
  CM is at L/2 ( halfway ) and put in the Force on a FBD
  2. The hinge changes everything!




$\boxed{a = \alpha\, L}$

### Example: Rotating Rod

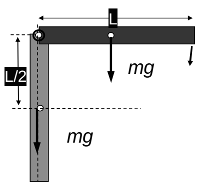
- A uniform rod of length L=0.5 m and mass m=1 kg is free to rotate on a frictionless hinge passing through one end as shown. The rod is released from rest in the horizontal position. What is

  (A) its angular speed when it reaches the lowest point ?

  1. For forces you need to locate the Center of Mass

  CM is at L/2 ( halfway ) and use the Work-Energy Theorem

  2. The hinge changes everything!




$W = mgh = \frac{1}{2} I \omega^2$

$W = mgL/2 = \frac{1}{2} (I_{CM} + m (L/2)^2) \omega^2$

and <u>solve</u> for $\omega$



### Connection with CM motion

- If an object of mass $M$ is moving linearly at velocity $V_{CM}$ *without* rotating then its kinetic energy is

$$K_T = \tfrac{1}{2} M V_{CM}^2$$

- If an object of moment of inertia $I_{CM}$ is rotating *in place* about its center of mass at angular velocity $\omega$ then its kinetic energy is

$$K_R = \tfrac{1}{2} I_{CM} \omega^2$$

- What if the object is both moving linearly and rotating?

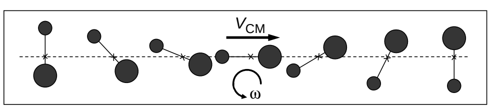
$$\boxed{K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M V_{CM}^2}$$



### Connection with CM motion...

- So for a solid object which rotates about its center of mass and whose CM is moving:

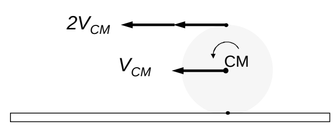
$$\boxed{K_{TOT} = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M V_{CM}^2}$$

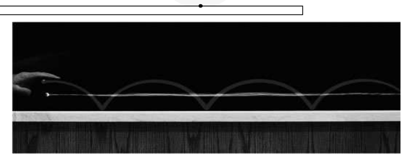





### Rolling Motion

- Again consider a cylinder rolling at a constant speed.



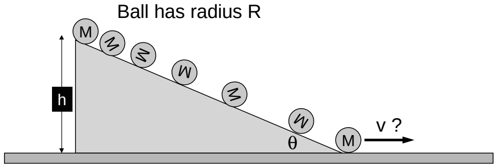





### Example : Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane ?

Ball has radius R

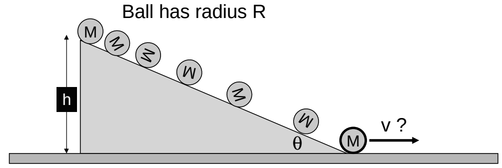





### Example : Rolling Motion

- A cylinder is about to roll down an inclined plane. What is its speed at the bottom of the plane ?
- Use Work-Energy theorem

Ball has radius R



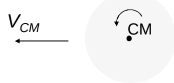

$Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} I_{CM}\, \omega^2$

$Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} (\frac{1}{2} M R^2)(v/R)^2 = \frac{3}{4} Mv^2$

$v = 2(gh/3)^{\frac{1}{2}}$

**Rolling Motion**

- Now consider a cylinder rolling at a constant speed.



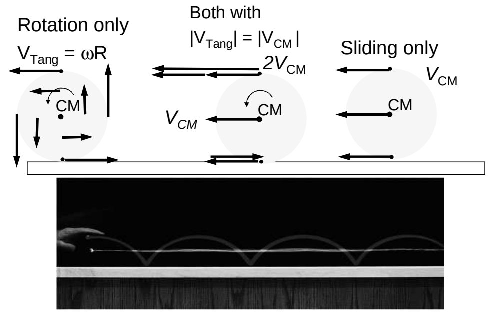

The cylinder is rotating about CM and its CM is moving at constant speed ($V_{CM}$). Thus its total kinetic energy is given by :

$$\boxed{K_{TOT} = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M V_{CM}^2}$$



## Motion

- Again consider a cylinder rolling at a constant speed.






**Angular Momentum:**

- We have shown that for a system of particles,

momentum $\vec{p} \equiv m\vec{v}$

is conserved if $\vec{F} = \frac{d\vec{p}}{dt} = 0$

- What is the rotational equivalent of this?

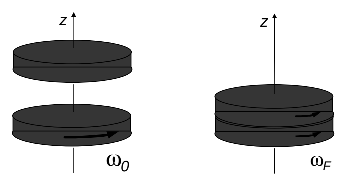
angular momentum $\vec{L} \equiv I\vec{\omega}$

is conserved if $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$



## Example: Two Disks

- A disk of mass *M* and radius *R* rotates around the *z* axis with angular velocity $\omega_0$. A second identical disk, initially not rotating, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_F$.



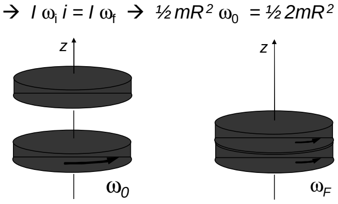



## Example: Two Disks

- A disk of mass *M* and radius *R* rotates around the *z* axis with initial angular velocity $\omega_0$. A second identical disk, at rest, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_F$.

*No External Torque so* $\mathbf{L}_z$ *is constant*

$L_i = L_f \rightarrow I\,\omega_i\, i = I\,\omega_f \rightarrow \frac{1}{2} mR^2\,\omega_0 = \frac{1}{2} 2mR^2\,\omega_f$






Lecture 16, Oct. 29

Assignment:

- Wednesday is an exam review session, Exam will be held in rooms B102 & B130 in Van Vleck at 7:15 PM
- MP Homework 7, Ch. 11, 5 problems,

  NOTE: Due Wednesday at 4 PM
- MP Homework 7A, Ch. 13, 5 problems, available soon

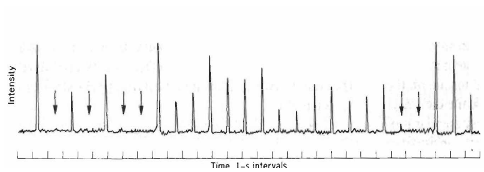
### An example: Neutron Star rotation

Neutron star with a mass of 1.5 solar masses has a diameter of ~11 km.

Our sun rotates about once every 37 days

$\omega_f / \omega_i = I_i / I_f = r_i^2 / r_f^2 = (7\text{x}10^5 \text{ km})^2/(11 \text{ km})^2 = 4 \text{ x } 10^9$

gives millisecond periods!




period of pulsar is 1.187911164 s



### Angular Momentum as a Fundamental Quantity

- The concept of angular momentum is also valid on a submicroscopic scale
- Angular momentum has been used in the development of modern theories of atomic, molecular and nuclear physics
- In these systems, the angular momentum has been found to be a fundamental quantity
  - ❖ Fundamental here means that it is an intrinsic property of these objects



### Fundamental Angular Momentum

- Angular momentum has discrete values
- These discrete values are multiples of a fundamental unit of angular momentum
- The fundamental unit of angular momentum is h-bar

❖ Where *h* is called Planck's constant

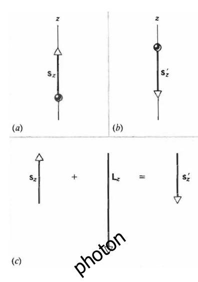
$$\hbar = \frac{h}{2\pi} = 1.054\times10^{-34}\ \frac{\text{kg}\cdot\text{m}}{\text{s}}$$

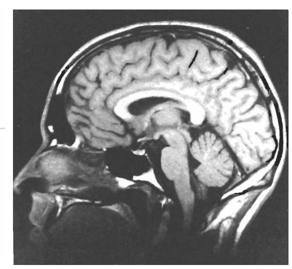
$$\Delta L = n\hbar \ (n = 1,2,3,...)$$



### Intrinsic Angular Momentum

intrinsic angular momentum of a proton is $\hbar/2$



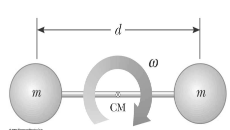





### Angular Momentum of a Molecule

- Consider the molecule as a rigid rotor, with the two atoms separated by a fixed distance
- The rotation occurs about the center of mass in the plane of the page with a speed of

$$\omega \approx \frac{\hbar}{I_{\text{CM}}}$$

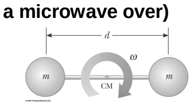





### Angular Momentum of a Molecule
### (It heats the water in a microwave over)

$$\Delta L = \hbar$$

$$\Delta\omega = \hbar / I_{\text{CM}}$$




E = $h^2/(8\pi^2 I)$ [ J (J+1) ]  J = 0, 1, 2, ….