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**Magnons**

| $\hbar \omega = 4J |S|(1 - \cos ka)$ | vs. | **Phonons** |
|----------------------------------|-----|-------------|
| $\omega^2 = \frac{2C}{\Lambda T}(1 - \cos ka)$ |

Notice that when $ka \approx 0 \quad \hbar \omega \approx 2JSk^2a^2 \propto k^2$
Title: Spin Waves (for a ferromagnetic system)

So far we have shown that the Coulomb force can give rise to an interaction in which parallel spins are the most energetically favorable. Thus the molecular field in the expression

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\[ |\vec{\mu}| = \frac{1}{2} g \mu_B \quad \vec{M} = \chi \vec{B} \quad \vec{\mu} \cdot \vec{H}_{\text{ex}} = U = -2(\sum_j J S_j) \cdot S_i \]

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